Mastermind Revisited

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Abstract

For integers n and k, we define $r(n,k)$ as the average number of guesses needed to solve the game of Mastermind for n positions and k colours; and define $f(n,k)$ as the maximum number of guesses needed. In this paper we add more small values of the two parameters, and provide exact value for the case of $n = 2$. Finally we comment on the asymptotics.

1 Introduction

In the original version of Mastermind (trademark), there are two players, CodeBreaker and CodeSetter. The CodeSetter creates a secret code of 4 pegs, each peg drawn from the same palette of 6 colours. The CodeBreaker must infer the secret code by asking a series of questions. Each question is itself a candidate code, and the response is two integers called Black and White. The value of Black indicates in how many positions there is exact agreement, and the value of White indicates how many colours are correct but in the wrong position. Another way to put this is that the sum of Black and White is the maximum number of blacks over all permutations of the guess code.

There is now an extensive literature on the game. This include papers on exact values (e.g. [4] for the original 6-colour 4-peg game), asymptotics (e.g. [1]), computer strategies based on calculation (e.g. [5]) and computer programs using artificial intelligence ideas (e.g. [2]).

In this paper we consider the exact values. For n positions and k colours, we define $r(n,k)$ (r for random) as the average number of guesses needed to solve the game where each secret code is equally likely. The most important result in this area is that of Koyama and Lai [4] who showed that the original 6-colour 4-position game has the solution $r(4,6) = 5625/1296$.

For n positions and k colours, we define $f(n,k)$ (f for foe) as the maximum number of guesses needed to solve the game against all codes. In a famous paper, Knuth [3] showed that the original 6-colour 4-position game can always be solved in 5 guesses, via a greedy strategy. In contrast, Chvátal [1] and Viaud [6] considered the asymptotics.

In this paper we add more small values of both $r(n,k)$ and $f(n,k)$. In particular we provide the exact value for $n = 2$ and all k. We also show that Knuth's strategy does not always solve the game in the shortest guaranteed number of moves. Finally, we add some comments on asymptotics.

2 Small Values

The following table gives the known values of $r(n,k)$:

Colours		Positions					
		2	3	4	5	6	7
	2	2	2.250	2.750	3.031	3.500	3.875
	3	2.333	2.704	3.037	3.358		
	4	2.813	3.219	3.535			
	5	3.240	3.608	3.941			
	6	3.667	3.954	4.340			
	7	4.041	4.297				
	8	4.438					
	9	4.790					
	10	5.170					

All the values except the $r(4,6)$ of Koyama and Lai are new, though the very small values must surely have been observed by several people. These formulas were determined by computer search (much like in [4]).

The following table gives the known values of $f(n,k)$. These were generated by computer search too.

Colours		Positions						
		2	3	4	5	6	7	8
	2	3	3*	4	4*	5*	5*	6*
	3	4	4	4	4	5	≤ 6	
	4	4	4	4	5	≤ 6		
	5	5	5	5	≤ 6			
	6	5	5	5				
	7	6	6	≤ 6				
	8	6						
	9	7						
	10	7						

For the values marked with a star, Knuth's strategy fails to be optimal. His strategy was: as next guess always take the code which minimises the maximum possible number of remaining contenders. For example, in the 2-colour 5-position game, his strategy would take 11112 as the first move, but actually need to start with 11122.

3 Two Positions

We use the usual device of writing a response as a two-digit number, where the first digit is the number of blacks and the second digit the number of whites.

Theorem 1 *For $k \geq 2$ the minimum number of guesses needed to guarantee solution of the 2-position k-color game is $f(2,k) = \lfloor k/2 \rfloor + 2$.*

PROOF.(a) A strategy. Guess $\lceil k/2 \rceil$ times using two new colours each time. There is a positive response at most twice. If a positive response twice, then it can easily be shown that there are at most two possible secrets. (For example if ab receives **10** and cd receives **01**, then secret is either ac or db.) If no positive response, then k is odd and the secret consists of the unused colour.

If a single positive, then the worst case is ab receives **10** and there is a missing colour x. Then the four possibilities are aa, bb, ax and xb, but these can be separated by guessing ax.

(b) A lower bound. Consider any $\lceil k/2 \rceil - 1$ guesses. At least two colours a and b are unused. The four codes are aa, bb, ab and ba, cannot be separated by one guess. QED

A general observation is that

$$f(n,k) \geq 1 + f(n-k,k),$$

since response of **0** to first guess leaves at least $n-k$ colours.

Theorem 2 *For $k \geq 2$, the average number of guesses needed to solve the 2-position k-color game is*

$$r(2,k) = k/3 + 17/8 + o(1).$$

The idea is to first solve the game for a set of code of the form aX (with X a set of colours), then for a$X \cup X$b and for a$X \cup X$b $\cup \{$aa, bb$\}$, and then solve the real game. One obtains a series of recurrence relations which are solved with help from a computer. We relegate the details to the appendix.

4 Asymptotics

There is a trivial lower bound based on the observation that: for n positions and k colours there are k^n codes but each question can reduce the possibilities by a factor of at most $\binom{n+2}{2}$. So:

$$\frac{n \log k}{\log \binom{n+2}{2}} \leq a(n,k) \leq f(n,k).$$

Chvátal [1] determined the order of $f(n,k)$ if the number of positions is large relative to the number of colours. In particular, for a fixed k, he showed that

$$f(n,k) \leq O\left(\frac{n \log k}{\log n}\right),$$

which matches the lower bound.

Chvátal [1] also showed that $f(n,n) \leq O(n \log n)$, and thus for k large relative to n

$$f(n,k) \leq O(k/n + n \log n).$$

Similar results were obtained by Viaud [6].

When k is fixed, since the value of $a(n,k)$ is sandwiched between the lower bound and $f(n,k)$, the asymptotics are the same. However, when n is fixed, the asymptotics are slightly different, as can easily be shown. For fixed n:

$$a(n,k) \sim \frac{k}{n+1}$$

As a strategy, one guesses codes with n new colours until all colours are determined. Whenever a positive response is received, one determines the colours and their multiplicities. By result from statistics, the expected number of guesses to determine all colours is approx $k/(n+1)$. After that we are left with a problem with at most n colours and n positions, which can be solve in $O(n \log n)$ steps by result of Chvátal mentioned earlier. The value $k/(n+1)$ is also a lower bound as is takes that many queries on average just to determine the colours.

References

[1] V. Chvátal, Mastermind, *Combinatorica* **3** (1983), 325–329.

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[5] E. Neuwirth, Some stratgies for Mastermind, *Z. Oper. Res. Ser. A-B* **26** (1982), B257–B278.

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5 Appendix: Proof of Theorem 2

Let X_m be a set of m colours and a, b distinct colours. Define $\mathcal{H}_m$ as the m codes $\mathsf{a}X_m$ (with possibly $\mathsf{a} \in X$): that is, all guesses with a in the first position and the other colour in X. Define $\mathcal{G}_m$ as the set of $2m$ codes $\mathsf{a}X_m \cup X_m\mathsf{b}$. Define $\mathcal{F}_m$ as the set of $2m+2$ codes $\mathcal{G}_m \cup \{\mathsf{aa}, \mathsf{bb}\}$. Finally, define $\mathcal{T}_m$ as all m^2 codes taken from X.

Define $H(m)$ (resp. $G(m)$, $F(m)$, $T(m)$) to be the total number of guesses to solve $\mathcal{H}_m$, (resp. $\mathcal{G}_m$, $\mathcal{F}_m$, $\mathcal{T}_m$).

For $\mathcal{H}_m$ there are only two reasonable guesses: of the form xy and ax. The code xy reveals the secret if the secret is ax or ay; otherwise $m-2$ possibilities remain for the other colour. The code ax can be the secret; otherwise $m-1$ possibilities remain for the other colour. Thus we get recurrence for $m \geq 2$:

$$H(m) = m + \min\{2 + H(m-2), H(m-1)\}.$$

This solves to

$$H(m) = m^2/4 + 3m/2 - 7/8 - (-1)^m/8, \quad H(1) = 1.$$

For $G(m)$ and $m \geq 2$, there are only two reasonable guesses up to symmetry: xy and ax. A positive response to xy leaves 4 possibilities in two pairs, and so 6 more guesses will solve those. A positive response to ax, other than **10**, reveals the code; while a response of **10** or **0** reduces to the problem of $\mathcal{H}_{m-1}$. So we get the recurrence for $m \geq 2$:

$$G(m) = 2m + \min\{1 + 2H(m-1)), 6 + G(m-2)\}.$$

This solves to

$$G(m) = m^2/2 + 4m - 13/4 + (-1)^m/4, \quad G(1) = 3.$$

For $F(m)$ and $m \geq 2$ there are four reasonable guesses up to symmetry: xy, ax, xa and aa. For xy, positive means 4 possibilities, in two pairs; negative leaves $\mathcal{F}_{m-2}$. For ax, responses **01** and **20** reveal the code; responses **10** and **0** both leave $\mathcal{H}_m$ (with a or b added to X). For xa, response **02** reveals code, response **10** leaves $\{\mathsf{xb}, \mathsf{aa}\}$, **0** leaves $\mathcal{H}_m$, **01** leaves $\mathcal{H}_{m-1}$. The code aa is worse than either of the above. It follows that for $m \geq 2$:

$$F(m) = 2m + 2 + \min\{6 + F(m-2),\ 1 + 2H(m),\ 4 + H(m) + H(m-1)\}.$$

which solves to

$$F(m) = (m^2 + 9m + 6)/2, \quad F(0) = 3,\ F(1) = 7,\ F(2) = 13.$$

For $T(m)$ there are only two guesses up to symmetry, and the guess xy is easily shown to be better. The responses: **0** leaves $\mathcal{T}_{m-2}$, **01** leaves $\mathcal{G}_{m-2}$, **10** leaves $\mathcal{F}_{m-2}$, **02** leaves yx. Thus for $m \geq 2$

$$T(m) = m^2 + T(m-2) + F(m-2) + G(m-2) + 1.$$

This solves to

$$T(m) = m^3/3 + 17m^2/8 - 77m/24 + m(-1)^m/8 + 39/16 - 7(-1)^m/16, \quad T(1) = 1,\ T(2) = 8.$$

Thus expected number of turns is asymptotically $k/3 + 17/8 + o(1)$.